

1.7 – Diagonal, Triangular, and Symmetric Matrices

A square matrix in which all the entries off the main diagonal are zero is called a **diagonal matrix**.

3. Find the product by inspection.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}$$

8. Find A^2 , A^{-2} , and A^k (where k is any integer) by inspection.

$$A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \frac{1}{36} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{25} \end{bmatrix}$$

$$A^k = \begin{bmatrix} (-6)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 5^k \end{bmatrix}$$

A square matrix in which all entries below the main diagonal are zero is an **upper triangular** matrix.

A square matrix in which all entries above the main diagonal are zero is a **lower triangular** matrix.

If a matrix is either upper triangular or lower triangular (or both), it is said to be **triangular**.

Let $A = [a_{ij}]$



Any non zero entries are here in an upper triangular matrix

Any non zero entries are here in a lower triangular matrix

Theorem 1.7.1 Properties of Triangular Matrices

- The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.
- The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- A triangular matrix is invertible if and only if its diagonal entries are all nonzero.
- The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Pf (a): (lower triangular)

Let $A = [a_{ij}]$ be a lower triangular matrix.

Then $a_{ij} = 0$ if $i < j$, that is, if the row number is less than the column number.

But $A^T = [a_{ij}]^T = [a_{ji}]$, and $a_{ji} = 0$ if $j > i$, that is, if the row number is greater than the column number. Thus A^T is upper triangular.

19. Determine by inspection whether the matrix is invertible.

$$\begin{bmatrix} 0 & 6 & -1 \\ 0 & 7 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $a_{11} = 0$, the matrix is not invertible (because it's triangular)

23. Find the diagonal entries of AB by inspection.

$$A = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$(AB)_{11} = -3 \quad (AB)_{22} = 5 \quad (AB)_{33} = -6$$

46. Prove: If the matrices A and B are both upper triangular or both lower triangular, then the diagonal entries of both AB and BA are the products of the diagonal entries of A and B .

Upper triangular, AB : $a_{i1} a_{i2} \dots a_{in}$ b_{1j}
 b_{2j}
 \vdots
 b_{nj}

Pf: $a_{i1} b_{1j} + a_{i2} b_{2j} \dots a_{in} b_{nj}$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ upper triangular matrices. Then $a_{ij} = 0$ and $b_{ij} = 0$ if $i > j$.

The ij^{th} entry of AB is $(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$.

If $i > k$ (or $k < i$), $a_{ik} = 0$

and if $k > j$ (or $j < k$), $b_{kj} = 0$.

[Thus, AB is upper triangular.]

$(AB)_{ii} = \sum_{k=1}^n a_{ik} b_{ki}$. But if $i \neq k$,

then $a_{ik} b_{ki} = 0 \Rightarrow (AB)_{ii} = \sum_{i=1}^n a_{ii} b_{ii}$. ✓

$\begin{bmatrix} 3 & 1 & a \\ b & c & d \\ 4 & 2 & 5 \end{bmatrix}$ is symmetric if $a = d$
 $b = 1$
 $c \in \mathbb{R}$
 $d = 2$

Definition 1: A square matrix A is said to be **symmetric** if $A = A^T$.

Theorem 1.7.2 Algebraic Properties of Symmetric Matrices

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

- a) A^T is symmetric.
- b) $A + B$ and $A - B$ are symmetric.
- c) kA is symmetric.

Theorem 1.7.3 The product of two symmetric matrices is symmetric

if and only if the matrices commute. *Based on $(AB)^T = B^T A^T$*

Theorem 1.7.4 If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Theorem 1.7.5 If A is an invertible matrix, then AA^T and $A^T A$ are also invertible.

*A invertible $\Rightarrow A^T$ is invertible (Thm 1.4.9).
Then AA^T and $A^T A$ are each products
of invertible matrices, which are invertible
by Thm 1.4.6.*